Exhaustive search methods for CNS polynomials

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Abstract

In this paper we present a method for finding all expansive polynomials with a prescribed degree n and constant term c_0. Our research is motivated by the fact that expansivity is a necessary condition for number system constructions. We use the algorithm for an exhaustive search of CNS polynomials for small values of

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n and c_0. We also define semi-CNS polynomials and show that producing them the same search method can be used.

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1 Introduction

Let Λ be a lattice in $\mathbb{R}^n$, $M : \Lambda \to \Lambda$ be a linear operator such that $\det(M) \neq 0$, and let D be a finite subset of Λ containing 0. The triple (Λ, M, D) is called a *number system* (or having the unique representation property) if every element x of Λ has a unique, finite representation of the form $x = \sum_{i=0}^{l} M^i d_i$, where $d_i \in D$ and $l \in \mathbb{N}$. The operator M is called the *base* or *radix*, D is the *digit set*. By a suitable basis transformation we may assume $\Lambda = \mathbb{Z}^n$. Let $c(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1} + x^n$ be a polynomial with integer coefficients and let M be its $n \times n$ companion matrix. Let furthermore $d_i = (i, 0, \ldots, 0) \in \mathbb{Z}^n$ and $D = \{\, d_i \mid 0 \leq i < |c_0| \,\}$. D is called a canonical digit set, and if (Λ, M, D) is a number system, then we call it a *canonical number system* or CNS. In this case the polynomial $c(x)$ is called a CNS polynomial. Alternative definitions can be found in [1], [4] or [5]. We remark that W.J. Gilbert [14] used the terminology radix representation instead of canonical number system.

The problem of characterizing CNS polynomials is still open. The linear case is trivial. Quadratic CNS polynomials were classified by I. Kátai and B. Kovács [15, 16] and independently by W.J. Gilbert [14]. Cubic and quartic CNS polynomials were investigated by S. Akiyama, H. Brunotte, A. Pethő [2], H. Brunotte [9], and K. Scheicher, J.M. Thuswaldner [24]. The general characterization seems to be hard. However, there are some important special cases. The following was discovered by B. Kovács [17] for irreducible polynomials, and generalized slightly by A. Pethő [22]. Let $c(x) = c_0 + c_1x + \ldots + c_{n-1}x^{n-1} + x^n$. If $c_0 \geq 2$, and $c_{n-1} \leq \cdots \leq c_1 \leq c_0$, and $c(x)$ is not divisible by a cyclotomic polynomial, then $c(x)$ is a CNS polynomial. S. Akiyama, A. Pethő [4], S. Akiyama, H. Rao [5] and K. Scheicher, J.M. Thuswaldner [24] showed characterization results under the "dominant" condition

$$c_0 > |c_1| + \cdots + |c_{n-1}|. \tag{1}$$

The second author investigated the case $c_0 = 2$ by computer [18] and gave all "binary" CNS polynomials up to the degree 8. This case has special interest since, via binary CNS, the existence of number systems with two digits can be characterized. To be more precise, let M be an expanding operator in $\mathbb{Z}^k$ with $|\det(M)| = 2$. Then there is a digit set

D for which $(\mathbb{Z}^k, M, D)$ is a number system if and only if M is $\mathbb{Z}$-similar to the companion matrix C_M of the characteristic polynomial of M, and $(\mathbb{Z}^k, C_M, \{0, d_1\})$ is a number system (Barbé, von Haeseler [6]).

It is known (see e.g. [19]) that one of the necessary conditions for the number system property is the expansivity of M. This means that a necessary condition for a polynomial to be a CNS polynomial is that it is expansive (or expanding) that is, all its complex roots lie outside the closed unit disk.

The paper is built up as follows. In section 2, we give an algorithm for finding all expansive polynomials with a fixed degree and constant term. In section 3, we define semi-CNS polynomials, a possible generalization of CNS polynomials. We also give some sufficient conditions for an expansive polynomial to be a semi-CNS polynomial, and briefly describe the decision algorithms used for the cases not covered by these conditions. In section 4, we give our computational results, and list some CNS and semi-CNS polynomials.

Below all polynomials have integer coefficients, unless otherwise stated.

2 Searching for expansive polynomials

In [18] the second author gave bounds on the coefficients of expansive polynomials which only depend on the degree and the constant term. His results can be stated in the following form.

Statement 2.1. *Let* $c_0 + c_1 x + \ldots + c_{n-1}x^{n-1} + x^n$ *be an expansive polynomial of degree* n*. Then*

$$|c_k| < \binom{n-1}{k-1} + |c_0| \cdot \binom{n-1}{k}, \qquad (1 \leq k \leq n).$$

These bounds were used to find all binary expansive polynomials of degree up to 8 by performing an exhaustive search in the region determined by the above inequality. Although these bounds are sharp for complex coefficients, the search revealed that expansive polynomials with integer coefficients are scarce in the region, and the bounds are far from sharp in the integer case. The authors' efforts on directly improving the bounds for integer coefficients failed. Since the size of the region grows in an over-exponential way with the degree, reaching beyond the degree 8 in the binary case needed optimization.

The authors revised the search algorithm, and managed to make several improvements. If one fixes some of the coefficients, they allow for better bounds on the unfixed ones, which resulted in an optimized search algorithm that found all binary expansive polynomials up to degree 11. A cluster of computers was used to go up to degree 12, for this grid, see [27]. The decision of expansivity was performed using a method described in [10].

In order to extend the search to larger constant terms and higher degrees, a different algorithm was applied. Prior application of the presented algorithm for finding CNS-polynomials is unknown to the authors, although similar algorithms already appeared in numeration research: finding Pisot and Salem numbers in intervals of the real line Boyd [7] used similar method. We only describe the algorithm, since the proofs of the related algorithms can be easily adapted for our case.

The method originated in two papers by Schur [25], [26], in which he examined power series of bounded holomorphic functions in the unit disk. His methods were generalized by Dufresnoy and Pisot [13] for meromorphic functions that have a single pole in the unit disk, and by Chamfy [12] for the case of several poles. In the description of the algorithm, we follow Boyd [7].

We will denote the reciprocal polynomial of a polynomial P by P^*. Take a monic integer polynomial $P(z) = c_0 + c_1 z + \cdots + z^n$. The key idea is to consider $P^*(z) = 1 + c_{n-1}z + \ldots + c_1 z^{n-1} + c_0 z^n$ and the quotient $f(z) = \pm P(z)/P^*(z)$, where the sign is chosen so that $f(0) > 0$. The quotient has modulus 1 on the unit circle, and has a power series

$u_0 + u_1 z + u_2 z^2 + \cdots$ in 0 with integer coefficients, with constant term $\pm c_0$. The search algorithm essentially works by giving lower and upper bounds on u_n that depend on $u_0, u_1, \ldots, u_{n-1}$. The following theorem holds.

Theorem 2.1. *Let $f(z) = \pm P(z)/P^*(z) = u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + u_n z^n + \cdots$, where P is monic expansive of degree at least n, and the sign is chosen so that $u_0 > 0$.*

- *There exists a unique monic expansive polynomial $Q(z)$ of degree n and a $v_n = v_n(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}$ so that the power series expansion of $Q(z)/Q^*(z)$ begins with $u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + v_n z^n$.*
- *There exists a unique monic expansive polynomial $R(z)$ of degree n and $w_n = w_n(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}$ with $-R(z)/R^*(z) = u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + w_n z^n + \cdots$.*

We have $v_n \leq u_n \leq w_n$. Equality holds on the left if and only if $P = Q$ and equality holds on the right if and only if $P = R$.

Given $u_0, u_1, \ldots u_{n-1}$, the coefficients of Q and R can be determined by a system of linear equations. For the calculations it is more convenient to use the following relations:

$$\begin{aligned} Q_{n+1}(z) &= (1+z)Q_n(z) - \frac{u_n - v_n}{u_{n-1} - v_{n-1}} z Q_{n-1}(z), \\ R_{n+1}(z) &= (1+z)R_n(z) - \frac{u_n - w_n}{u_{n-1} - w_{n-1}} z R_{n-1}(z). \end{aligned}$$

For the proof of the theorem and the recurrence relations we refer to [13] or [12], where analogous statements are proved.

Fix an integer u_0. Using the theorem, one can build a rooted tree of Taylor-polynomials. The root is u_0, and the children of a node $(u_0 + u_1 z + \cdots + u_{n-1} z^{n-1})$ are $(u_0 + u_1 z + \cdots + u_n z^n)$ for $v_n(u_0, u_1, \ldots, u_{n-1}) \leq u_n \leq w_n(u_0, u_1, \ldots, u_{n-1})$, $u_n \in \mathbb{Z}$, if $v_n < w_n$. If $v_n \geq w_n$, then the node is a leaf. This tree can be built using the recurrence relations above, and traversed to any depth n using a tree traversal algorithm (the authors used depth first search). Every monic expansive polynomial $P(z) \in \mathbb{Z}[z]$ of degree at most n and constant term $\pm u_0$ can be found by solving $P(z)/P^*(z) = f(z)$ for some leaf f of the tree up to level n.

We illustrate the algorithm for the binary case in figure 1. The nodes of the tree are polynomials of form $u_0 + u_1 z + \cdots + u_n z^n$ that are truncations of the power series of P/P^* for some P monic expansive polynomial with integer coefficients and constant term $\pm u_0$. Expansive polynomials of degree n can be found by looking at leaves at level n. Linear and quadratic expansive polynomials P are shown in parentheses below the

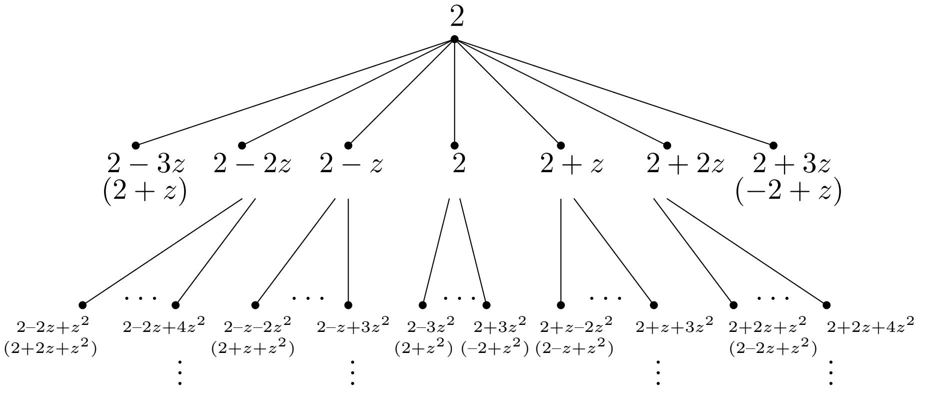

Figure 1: The first two levels of the tree of Taylor-polynomials, $u_0 = 2$.

corresponding leaves P/P^*.

We observed that when c_0 becomes large, the tree becomes extremely wide, which slows down the algorithm. Although asymptotically (with the degree) the tree traversal algorithm performs faster than the one used in [18], empirical data suggest that for large c_0 and small degrees the original algorithm is faster.

3 CNS and semi-CNS polynomials

3.1 CNS-polynomials

Given an expansive polynomial, there are several ways of deciding if it is a CNS polynomial or not. We used Brunotte's algorithm and an enhanced version of a method that performs an exhaustive search for possible cycles in a region. The detailed description of these decision algorithms can be found in [11]. Experiments show that the time complexity of both decision algorithms grows rapidly with the degree, and for Brunotte's algorithm this is also true for space complexity. It can be observed however that negative answers are obtained quickly by both algorithms.

Let $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + x^n$ be given. If the dominant condition (1) holds then there are sets of conditions, by which $c(x)$ is CNS. For $n = 3$ and $n = 4$ see [5, 24], for $n = 5$ and higher see [5]. Under the dominant condition both of the mentioned decision algorithms are fast. Moreover, there are other known methods as well ([4], [5], Theorem 4.3, Corollary 4.4). Without the dominant condition we know the following results:

- The theorem of B. Kovács, proved in [17], was already mentioned in the introduction.

- H. Brunotte characterized CNS-trinomials in [8]. Let $n > 2$.

  (i) The polynomial $x^n + bx + c$ is a CNS polynomial if and only if $-1 \le b \le c - 2$.

  (ii) Let $1 < q < n$, $q \nmid n$. The polynomial $x^n + bx^q + c$ is a CNS polynomial if and only if $0 \le b \le c - 2$.

The next theorem shows that CNS polynomials of degree n can be "lifted" to yield polynomials of degree nk. This is also proved by Brunotte [8] in a different way.

Theorem 3.1. *Let $c(x) = c_0+c_1x+c_2x^2+\cdots+c_nx^n$ be a CNS polynomial. Then, for any integer $k \geq 1$, $c(x^k) = c_0 + c_1x^k + c_2x^{2k} + \cdots + c_nx^{nk}$ is also a CNS polynomial.*

Proof. Identify $\mathbb{Z}^{nk}$ with the quotient ring $\mathbb{Z}[x]/c(x)\mathbb{Z}[x]$. Every element p of this ring can be written in the form

$$p = \sum_{i=0}^{nk-1} a_i x^i = \sum_{i=0}^{k-1} x^i \sum_{j=0}^{n-1} a_{kj+i} x^{kj+i} .$$

¿From the CNS property of $c(x)$ one has the radix expansions

$$\sum_{j=0}^{n-1} a_{kj+i} x^j = \sum_{j=0}^{l_i} d_j^{(i)} x^j \quad 0 \leq i \leq k-1 ,$$

where $l_i \in \mathbb{N}$ and $d_j^{(i)}$ are digits. But then

$$p = \sum_{i=0}^{k-1} \sum_{j=0}^{l_i} d_j^{(i)} x^{kj+i}$$

is a radix expansion for $c(x^k)$. Unicity of the expansion can be proved analogously. □

3.2 Semi-CNS polynomials

It is well-known and easy to show that a monic polynomial with negative constant term cannot be a CNS polynomial. A classical example is $x-10$, meaning that the ordinary decimal number system is not a CNS, since negative numbers have no representation. We introduce the following definition:

Definition 3.2. *Let c_0 be an integer. A polynomial $P(x) = c_0+c_1x+\cdots+x^n$ is called a semi-CNS polynomial if for the digit set $D = \{0, 1, \ldots |c_0|-1\}$ the finite expansions $\left\{\sum_{i=0}^{l} d_i x^i \mid l \in \mathbb{N}, d_i \in D\right\}$ form an additive semigroup.*

With this definition, $x-c_0$ becomes a semi-CNS polynomial for $c_0 > 1$. With a small modification of Brunotte's algorithm ([5],[9]) it is easy to decide the semi-CNS property.

Suppose that D is a complete residue system for an expansive $n \times n$ matrix M. For $p \in \mathbb{Z}^n$, we denote with $\tau(p)$ the unique element $q \in \mathbb{Z}^n$ for which there exists $d \in D$ so that $p = Mq + d$.

Theorem 3.3. *(Brunotte's algorithm for semi-CNS) Let $c(x)$ be an expansive polynomial, M its companion matrix and D the canonical digit set. Let τ be the above map. If there exists a set $E \subseteq \mathbb{Z}^n$ such that*

(i) $(1, 0, \ldots, 0) \in E$,

(ii) *For all $e \in E$ and $d \in D$ we have $\tau(e + d) \in E$,*

(iii) *For all $e \in E$ there exists a positive integer k so that $\tau^k(e) = 0$,* *then $c(x)$ is a semi-CNS polynomial.*

The proof is identical to Brunotte's proof in [9]. Note that for the CNS property $(\pm 1, 0, \dots, 0) \in E$ is needed in (i). The construction of such a set E (or the proof of its non-existence) is usually done by choosing an initial set E_0, and enlarging it until it satisfies (ii). Then one checks whether (iii) holds. Below we give some properties of semi-CNS polynomials.

It is noted in [3] that for CNS candidate polynomials the final set E is the same for $E_0 = \{(1, 0, \dots, 0)\}$ and $E_0 = \{(\pm 1, 0, \dots, 0)\}$. This means that when the constant term of a polynomial is positive, the semi-CNS and CNS properties are equivalent.

For negative constant terms one has the following condition.

Theorem 3.4. *Let $c_0 + c_1 x + \dots + c_{n-1}x^{n-1} + x^n$ be an expansive polynomial with $c_0 < 0$. If no other coefficient is negative, then it is a semi-CNS polynomial.*

Proof. Expansivity implies $c_1 + c_2 + \cdots c_{n-1} + 1 < |c_0|$, otherwise there would be a real root between 0 and 1. Using Brunotte's basis (see [9]), the set $E = \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \mid \varepsilon_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq n\}$ satisfies the conditions of theorem 3.3. □

Statement 3.1. *Let k, n be positive integers. The number of polynomials* *of degree n and constant term $-k$ satisfying the conditions of theorem 3.4 is $\binom{n+k-3}{k-2}$.*

Proof. It is easy to see that $c_1 + c_2 + \cdots c_{n-1} + 1 < |c_0| = k$ is also a sufficient condition of expansivity. We have to determine the number of non-negative tuples $(c_1, c_2, \ldots, c_{n-1})$ that sum up to at most $k-2$. That is equal to the number of ordered partitions of $k-2$ into n parts, the expression above. □

4 Computational Results

The search algorithm was implemented in C/C++, using multi-precision arithmetic. It can be parallelized using the inherent parallel nature of the algorithm. Parts of the search were performed on the Desktop Grid of the Hungarian Academy of Sciences [27, 20]. The decision algorithms were implemented in C++, much attention being paid for memory-efficiency in the case of Brunotte's algorithm. They were performed on desktop computers.

Since the results for degrees up to 8 were known earlier ([18]), we extend this list with the binary CNS polynomials for degrees 9, 10 and 11.

There are altogether 192 expansive polynomials of degree 9 with constant term 2. The following 12 of them are CNS polynomials: $2-x+x^9$, $2-x^3+x^9$, $2+x^9$, $2+x^4+x^5+x^9$, $2+x^3+x^6+x^9$, $2+2x^3+2x^6+x^9$, $2+x^2+x^7+x^9$, $2+x+x^8+x^9$, $2+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9$, $2+2x+2x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+x^7+x^8+x^9$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+2x^7+2x^8+x^9$.

There are altogether 623 expansive polynomials of degree 10 with constant term 2. The following 42 of them are CNS polynomials: $2-x+x^{10}$, $2-x^2+x^{10}$, $2-x^2+x^4+x^{10}$, $2-x^5+x^{10}$, $2+x^{10}$, $2+x^5+x^{10}$, $2+2x^5+x^{10}$, $2+x^4+x^6+x^{10}$, $2+x^3+x^7+x^{10}$, $2+x^2+x^8+x^{10}$, $2+x^2+x^4-x^5+x^6+x^8+x^{10}$, $2+x^2+x^4+x^6+x^8+x^{10}$, $2+x^2+x^4+x^5+x^6+x^8+x^{10}$, $2+x^2+x^3+2x^5+x^7+x^8+x^{10}$, $2+2x^2+2x^4+2x^6+2x^8+x^{10}$, $2+x+x^9+x^{10}$, $2+x+x^4+2x^5+x^6+x^9+x^{10}$, $2+x+x^2+x^3+x^4+x^6+x^7+x^8+x^9+x^{10}$, $2+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+x+x^2+x^3+x^4+2x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+x+2x^2+x^3+2x^4+x^5+2x^6+x^7+2x^8+x^9+x^{10}$, $2+2x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+2x+x^2+2x^3+2x^4+x^5+2x^6+2x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+x^3+2x^4+2x^5+2x^6+x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+2x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+2x^4+2x^5+x^6+x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+2x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+3x^3+3x^4+2x^5+2x^6+$

$2x^7+x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+2x^7+2x^8+x^9+x^{10}$, $2+2x+3x^2+2x^3+3x^4+2x^5+3x^6+2x^7+2x^8+x^9+x^{10}$, $2+2x+3x^2+3x^3+3x^4+3x^5+3x^6+2x^7+2x^8+x^9+x^{10}$, $2+2x+3x^2+3x^3+4x^4+3x^5+3x^6+2x^7+2x^8+x^9+x^{10}$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+2x^7+2x^8+2x^9+x^{10}$, $2+3x+3x^2+3x^3+3x^4+3x^5+3x^6+3x^7+3x^8+2x^9+x^{10}$, $2+3x+4x^2+4x^3+4x^4+4x^5+4x^6+4x^7+3x^8+2x^9+x^{10}$, $2+3x+4x^2+5x^3+5x^4+5x^5+5x^6+4x^7+3x^8+2x^9+x^{10}$, $2+3x+4x^2+5x^3+6x^4+6x^5+5x^6+4x^7+3x^8+2x^9+x^{10}$, $2+4x+5x^2+5x^3+5x^4+5x^5+5x^6+4x^7+3x^8+2x^9+x^{10}$. $2+4x+6x^2+7x^3+7x^4+6x^5+5x^6+4x^7+3x^8+2x^9+x^{10}$.

There are altogether 339 expansive polynomials of degree 11 with constant term 2. The following 11 of them are CNS polynomials: $2-x+x^{11}$, $2+x^{11}$, $2+x^5+x^6+x^{11}$, $2+x^4+x^7+x^{11}$, $2+x^3+x^8+x^{11}$, $2+x^2+x^9+x^{11}$, $2+x+x^{10}+x^{11}$, $2+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}$, $2+2x+2x^2+2x^3+2x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+x^7+x^8+x^9+x^{10}+x^{11}$, $2+2x+2x^2+2x^3+2x^4+2x^5+2x^6+2x^7+2x^8+2x^9+2x^{10}+x^{11}$.

There are 1085 expansive polynomials of degree 12 with constant term 2. The number of CNS polynomials among them is between 56 and 66. For degree 13 the number of CNS polynomials is between 14 and 15 out of 526 expansives, and for degree 14 the number is between 29 and 45 out of 1283. The authors find it very likely that the precise answers are 66,

c_0\Degree	2	3	4	5	6	7	8
2	5/4	7/4	29/12	29/7	105/25	95/12	309/32
3	7/5	25/13	131/47	310/75	1413/242	2619/322	10273/816
4	9/6	51/26	327/108	1240/286	6749/1033	20129/2194	
5	11/7	85/43	655/200	3369/735	21671/3010		
6	13/8	127/63	1155/332	7468/1546	55785/7106		
7	15/9	177/88	1829/509	14411/2876	122633/14606		
8	17/10	235/115	2747/742	25265/4887	241391/27263		
9	19/11	301/147	3905/1025	41331/7802			
10	21/12	375/182	5379/1378	63959/11824			

Table 1: The number of expansive and CNS polynomials, ordered by degree and constant term. (The missing cells are being computed.)

14 and 45. The uncertainty is due to the fact that the set of witnesses in Brunotte's algorithm becomes very large and does not fit into computer memory.

In tables 1 (resp. 2) we list the number of expansive/CNS (resp. semi-CNS) polynomials by degree n and constant term c_0 (resp. $-c_0$).

5 Summary

We presented a fast algorithm for finding all monic expansive polynomials with fixed degree and constant term. In our future work we wish to extend the results shown in tables 1 and 2. We also plan to examine the obtained expansive polynomials for number system property using symmetrical digit set and other kinds of non-canonical digit sets.

$-c_0\backslash$Degree	2	3	4	5	6	7	8
2	1/1	7/1	7/1	29/1	23/1	95/1	57/1
3	3/2	25/3	55/4	310/5	563/6	2619/7	4091/8
4	5/3	51/6	179/10	1240/15	3605/21	20129/28	
5	7/4	85/10	421/20	3369/35	13501/56		
6	9/5	127/15	795/35	7468/70	37853/126		
7	11/6	177/21	1353/56	14411/126	88501/252		
8	13/7	235/28	2099/84	25265/210	182235/462		
9	15/8	301/36	3083/120	41331/330			
10	17/9	375/45	4349/165	63959/495			

Table 2: The number of expansive and semi-CNS polynomials, ordered by degree and constant term. (The missing cells are being computed.)

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